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EXACT SOLUTIONS FOR EVOLUTIONARY SUBMODELS OF GAS DYNAMICS

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A new class of exact solutions with functional arbitrariness describing the motion of a polytropic gas is constructed on the basis of invariant submodels of rank two of the evolutionary type. In the solutions obtained, the velocity is a linear function of some spatial coordinates. These solutions describe continuous gas dispersion and the motion with density collapse at a finite time.

Introduction. Invariant submodels of rank two for gas-dynamic equations of the evolutionary type describe exact solutions with two independent variables: time and a certain combination of spatial coordinates. These submodels are specified by a closed system of differential equations in partial derivatives. The desired functions in these submodels are invariant components of the velocity vector, density, and pressure. For gas-dynamic equations with an arbitrary equation of state there are ten different invariant submodels of the type studied, in particular, the submodel of one-dimensional gas motion. Mamontov constructed canonical forms of these ten submodels [1] and studied their group properties [2].

However, specific solutions of these submodels have been studied insufficiently. To reveal special features of every model, we chose a class of solutions with a linear dependence of one of the invariant velocity components on the invariant independent variable. Such solutions are classical for the submodel of one-dimensional gas motion [3]. The solutions of the other nine submodels can be characterized as follows. In all cases, integration yields an ordinary second-order differential equation. In the initial "physical" variables, these solutions correspond to motions with a linear dependence of the velocity component on certain spatial variables (in the general case, these motions were studied in [4]). The solutions describe continuous gas dispersion, density collapse at a finite time, and oscillatory motion. Below, trajectories of individual particles on the solutions obtained are described, which leads to a better understanding of the motion described by the submodels. All solutions are divided into two main classes: motions with particle trajectories in the form of flat curves and twisted motions. A more detailed description of these classes is given below.

1. Preliminary Data. Gas-dynamic equations are written in standard terms of the velocity vector $\boldsymbol{u} = (u, v, w)$, pressure p, and density ρ. All functions depend on the spatial coordinates $\boldsymbol{x} = (x, y, z)$ and time t. The functions $A(p, \rho)$ specify the equation of state for a gas:

$$D\boldsymbol{u} + \rho^{-1}\nabla p = 0, \quad D\rho + \rho \operatorname{div} \boldsymbol{u} = 0, \quad Dp + A(p, \rho) \operatorname{div} \boldsymbol{u} = 0,$$
$$D = \partial_t + u\,\partial_x + v\,\partial_y + w\,\partial_z. \tag{1.1}$$

It is known that all invariant evolutionary submodels of rank two for gas-dynamic equations can be written in canonical form [5]

$$U_t + UU_\lambda + b(t)\rho^{-1}p_\lambda = a_1, \quad V_t + UV_\lambda = a_2, \quad W_t + UW_\lambda = a_3,$$
$$\rho_t + U\rho_\lambda + \rho U_\lambda = \rho a_4, \qquad p_t + Up_\lambda + A(p, \rho)U_\lambda = A(p, \rho)a_4. \tag{1.2}$$

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Here $\boldsymbol{U} = (U, V, W)$ is the invariant velocity vector and t and λ are invariant independent variables. The right sides $(a_1, \ldots, a_4)$ depend on t, λ, and $\boldsymbol{U}$. The quantities λ and $\boldsymbol{U}$ and the right sides $a_1, \ldots, a_4$ are expressed in terms of initial variables in each submodel [1]. Mamontov [2] analyzes the group characteristics of all the submodels studied.

Below, we describe in detail a class of solutions of submodels (1.2) in which the component U of the invariant velocity $\boldsymbol{U}$ depends linearly on the variable λ. A structural description of this class of solutions is possible if the equation of state of a polytropic gas is chosen.

2. Integration of the Submodels. In Eqs. (1.2), we convert to the Lagrangian coordinates t and ξ (ξ is an arbitrary function satisfying the equation $\xi_t + U\xi_\lambda = 0$). In conversion to the Lagrangian variables using the function $M = \partial\lambda/\partial\xi$, the derivatives are transformed as follows: $h_t + Uh_\lambda \to h_t$ and $h_\lambda \to M^{-1}h_\xi$. The velocity component in the new variables is $U = \lambda_t$. Writing Eq. (1.2) in the Lagrangian variables, we have

$$\begin{gathered} \lambda_{tt} + b(t)p_\xi/(\rho M) = \tilde{a}_1, \qquad V_t = \tilde{a}_2, \qquad W_t = \tilde{a}_3, \\ \rho_t + M^{-1}\rho M_t = \rho\tilde{a}_4, \qquad p_t + M^{-1}A(p,\rho)M_t = A(p,\rho)\tilde{a}_4. \end{gathered} \tag{2.1}$$

The right sides $\tilde{a}_1, \ldots, \tilde{a}_4$ are functions of $a_1, \ldots, a_4$ in the Lagrangian variables. It is further assumed that the gas satisfies the polytropic equation of state $A(p, \rho) = \gamma p$ (γ is the adiabatic exponent). From Eqs. (2.1), it follows that

$$(\ln \rho M)_t = \tilde{a}_4, \qquad (\ln pM^\gamma)_t = \gamma\tilde{a}_4. \tag{2.2}$$

For all the nine submodels, the second and third equations in (2.1) and Eq. (2.2) with specific functions on their right sides are integrated explicitly. Below, we assume that the dependence $U(\lambda)$ is linear. In the Lagrangian variables, this means that $M = M(t)$:

$$\lambda = M(t)\xi, \qquad U = \dot{M}\xi. \tag{2.3}$$

Here and below, the dot and the prime denote differentiation of functions that depend only on t and ξ, respectively.

Remark 1. *The inhomogeneous dependence $\lambda = M(t)\xi + h(t)$ reduces to a homogeneous one because in all submodels, we can obtain $h(t) \equiv 0$ by action of a group of continuous transformations admitted by* (1.1).

Substituting (2.3) into the first equation in (2.1), we have the key equation

$$\ddot{M}\xi + b(t)\rho^{-1}M^{-1}p_\xi = \tilde{a}_1. \tag{2.4}$$

Substitution of the functions V, W, p, and ρ determined from (2.1) and (2.2) into Eq. (2.4) yields the structure

$$\boldsymbol{a}(t) \cdot \boldsymbol{b}(\xi) = 0 \tag{2.5}$$

with the vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ ($n = 2$, 3, and 4). We note that equations of the form of (2.5) are typical of the problems related to overdetermined systems of differential equations (group classification, partly invariant solutions, differential constraints, and *a priori* assumptions on the type of solution). Studying a specific relation of the problem of group classification of gas-dynamic equations, Ovsyannikov [6] proposed an idea that made it possible to separate the variables in (2.5). In the general form, the lemma on separation of variables in Eq. (2.5) is published here for the first time with consent of its author.

Ovsyannikov's Lemma. *Relation* (2.5) *holds if and only if there exist a number k ($0 \leqslant k \leqslant n$), partitions of the vectors $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'')$ and $\boldsymbol{b} = (\boldsymbol{b}', \boldsymbol{b}'')$ ($\boldsymbol{a}', \boldsymbol{b}' \in \mathbb{R}^k$ and $\boldsymbol{a}'', \boldsymbol{b}'' \in \mathbb{R}^{n-k}$), and a constant ($(n-k) \times k$) matrix C such that*

$$\boldsymbol{a}'' = C\boldsymbol{a}', \qquad \boldsymbol{b}' = -C^{\mathrm{t}}\boldsymbol{b}'', \tag{2.6}$$

where C^{t} is the transposed matrix C.

Proof. Let the linear capsule $\mathcal{L}\{\boldsymbol{a}(t)\}$ have dimension k and $\boldsymbol{a} = (a^1, \ldots, a^n)^{\mathrm{t}}$. Then, there exist k points t_j ($j = 1, \ldots, k$) such that the ($n \times k$) matrix $M = (a^i(t_j))$ (i is the row number) has rank k. We assume that the matrix rank minor is specified by a ($k \times k$) matrix A ($\det A \neq 0$), and the remaining rows form the ($(n-k) \times k$) matrix B. Let $\boldsymbol{a}'$ be the rows of the matrix M that form A and $\boldsymbol{a}''$ are the remaining rows. Addition of the row $(a^i(t))$ to the matrix M does not change the matrix rank. Therefore, there exists a vector $\boldsymbol{r} = \boldsymbol{r}(t) \in \mathbb{R}^k$ for which

$$\boldsymbol{a}' = A\boldsymbol{r}, \qquad \boldsymbol{a}'' = B\boldsymbol{r}$$

and, hence, $\boldsymbol{r} = A^{-1}\boldsymbol{a}'$. From this, we have the first relation of (2.6) with the ($(n-k) \times k$) matrix $C = BA^{-1}$. Then, (2.5) becomes $\boldsymbol{a}' \cdot \boldsymbol{b}' + (C\boldsymbol{a}') \cdot \boldsymbol{b}'' = 0$ or $\boldsymbol{a}' \cdot (\boldsymbol{b}' + C^{\mathrm{t}}\boldsymbol{b}'') = 0$, from which, by virtue of $\dim \mathcal{L}(\boldsymbol{a}') = k$, we have (2.6). Sufficiency of (2.6) is obvious.

When solving classification problems, we have to exhaust various values of $k = 0, \dots, n$, partitions of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and matrices C with indefinite elements.

Below, the submodels are divided into three types, depending on the dimension of the admitted group of continuous transformations. Submodels 2.21, 2.22, 2.24, and 2.25 belong to the first type (enumeration same as in [1]). The infinite-dimensional part of the admitted group contains two arbitrary functions of a Lagrangian variable. The second type includes submodels 2.8, 2.9, and 2.10, in which the admitted group contains one arbitrary function. Finally, submodels 2.20 and 2.23 admit only a finite-dimensional group. It is found that the initial formal division of the submodels into types is supported by similarity between physical characteristics of the solutions obtained.

3. Submodels of the First Type. *Submodel* 2.21. Golovin [7] analyzed this submodel. The solution representation is

$$u = U, \quad v = \frac{V + z + t(W + y)}{t^2 + 1}, \quad w = \frac{-W - y + t(V + z)}{t^2 + 1}, \quad \lambda = x.$$

In (1.2), $b = 1$, $a_1 = a_2 = a_3 = 0$, and $a_4 = -2t/(t^2 + 1)$. From the second and third equations in (2.1) and from Eq. (2.2), we have

$$V = V_0(\xi), \qquad W = W_0(\xi), \qquad \rho = \frac{f(\xi)}{M(t^2 + 1)}, \qquad p = \frac{P(\xi)}{M^\gamma (t^2 + 1)^\gamma}. \tag{3.1}$$

Here and below, V_0, W_0, P, and f are arbitrary functions. Separating the variables of (2.4) using the Lemma, we obtain the equation for M

$$\ddot{M} M^\gamma (t^2 + 1)^{\gamma - 1} = \alpha \qquad (\alpha = \text{const}) \tag{3.2}$$

similar to the Emden–Fowler equation and the following constraint on the functions P and f: $P'(\xi) = \alpha \xi f(\xi)$. The trajectory of a particle, starting at the initial time $t_0 = 0$ from a point with coordinates (x_0, y_0, z_0), is determined by the formulas

$$x = x_0 M(t), \qquad y = y_0 + t z_0 + t V_0(x_0), \qquad z = z_0 - t y_0 - t W_0(x_0).$$

Here and below, we use arbitrariness in the choice of α to satisfy the initial condition $M(t_0) = 1$.

Submodel 2.22. The solution representation is

$$u = U, \qquad v = V + y/t, \qquad w = W + z/t, \qquad \lambda = x.$$

For this submodel, in (1.2), we obtain $b = 1$, $a_1 = 0$, $a_2 = -V/t$, $a_3 = -W/t$, and $a_4 = -2/t$. Integration of Eqs. (2.1) taking into account (2.2) yields

$$V = V_0(\xi)/t, \quad W = W_0(\xi)/t, \quad \rho = f(\xi)/(M t^2), \quad p = P(\xi)/(M^\gamma t^{2\gamma}). \tag{3.3}$$

For the function M, we have the generalized Emden–Fowler equation

$$\ddot{M} M^\gamma t^{2\gamma - 2} = \alpha \qquad (\alpha = \text{const}). \tag{3.4}$$

The constraint on P and f is the following: $P'(\xi) = \alpha \xi f(\xi)$. Equation (3.4) has a particular solution $M = t^k$ with $k = (4 - 2\gamma)/(1 + \gamma)$. The particle trajectories are defined by the following formulas (here $t_0 = 1$):

$$x = x_0 M(t), \quad y = t(y_0 + V_0(x_0)) - V_0(x_0), \quad z = t(z_0 + W_0(x_0)) - W_0(x_0). \tag{3.5}$$

Submodel 2.24. The solution representation is

$$u = V + (x - \beta y)/t, \qquad v = W, \qquad w = U, \qquad \lambda = z.$$

For this submodel, in (1.2) $b = 1$, $a_1 = 0$, $a_2 = -V/t + \beta W/t$, $a_3 = 0$, and $a_4 = -1/t$. Integration of (2.1) taking into account (2.2) yields

$$V = V_0(\xi)/t + \beta W_0(\xi), \quad W = W_0(\xi), \quad \rho = f(\xi)/(M t), \quad p = P(\xi)/(M^\gamma t^\gamma). \tag{3.6}$$

The equation for M has the form

$$\ddot{M} M^\gamma t^{\gamma - 1} = \alpha \qquad (\alpha = \text{const}). \tag{3.7}$$

The constraint on P and f is the following: $P'(\xi) = \alpha \xi f(\xi)$. Equation (3.7) has a particular solution $M = t^k$ with $k = (3 - \gamma)(1 + \gamma)^{-1}$. The particle trajectories are defined by the following formulas ($t_0 = 1$):

$$x = t x_0 + (1 - t)(\beta y_0 - V_0(z_0)), \quad y = y_0 - (1 - t) W_0(z_0), \quad z = z_0 M(t).$$

Submodel 2.25. The solution representation is

$$u = V + z, \qquad v = U, \qquad w = W, \qquad \lambda = y.$$

For this submodel, in (1.2), $b = 1$, $a_1 = 0$, $a_2 = -W$, and $a_3 = a_4 = 0$. Integration of (2.1) taking into account (2.2) yields

$$V = V_0(\xi) - tW_0(\xi), \qquad W = W_0(\xi), \qquad \rho = f(\xi)/M, \qquad p = P(\xi)/M^\gamma. \tag{3.8}$$

The equation for M has the form

$$\ddot{M}M^\gamma = \alpha \qquad (\alpha = \text{const}). \tag{3.9}$$

The constraint on P and f is the following: $P'(\xi) = \alpha\xi f(\xi)$. Equation (3.7) has a particular solution $M = t^{2/(1+\gamma)}$. The equations of particle trajectories have the following form ($t_0 = 0$):

$$x = x_0 + t(z_0 + V_0(y_0)), \qquad y = y_0 M(t), \qquad z = z_0 + tW_0(y_0).$$

4. Description of Motion. In the case of motion of the first type, particle trajectories are plane curves. The slope and position of the plane in space are defined by the initial position of the particle. Motions in different planes are similar and defined by the form of the function $M(t)$. If $M(t) \neq 0$, the trajectories of various particles do not intersect. In contrast, if $M(t_*) = 0$ is satisfied for certain t_*, the collapse occurs: the density increases to infinity, and all particles arrive at the plane $\lambda = 0$.

The motion of gas particles in the aggregate can easily be treated as follows. Let us consider a layer of gas bounded by an immovable flat wall $\lambda = 0$ and a moving piston $\lambda_p = \lambda_0 M(t)$. The pressure on the piston depends only on time and varies by the law described by formulas (3.1), (3.3), (3.6), and (3.8) with $P(\xi) = P(\lambda_0) = \text{const}$. If $M(t)$ increases at $t > t_0$, the piston retreats from the wall $\lambda = 0$, and the gas rarefies. As $M(t) \to 0$, the piston approaches the wall, which leads to an unlimited increase in density and pressure. The gas layer evolves as follows. In addition to uniform rarefaction or compression in the λ direction determined by the function $M(t)$, the gas layer undergoes changes in a plane that is orthogonal to λ. Thus, in submodel 2.21, the entire gas layer rotates counter-clockwise around the λ axis through angle arctan t (relative to the coordinate origin) and dilates under the law $r = r_0\sqrt{t^2 + 1}$ [r is the distance from the point to the λ axis; $r_0 = r(t_0)$]. In the remaining submodels, rotation is absent. In submodel 2.22, the gas dilates in the plane $\lambda = \text{const}$ along the straight lines intersecting the λ axis; in submodels 2.24 and 2.25, the gas dilates along the positive Ox direction linearly in time. The functions $V_0(\xi)$ and $W_0(\xi)$ define the relative displacement of the planes $\lambda = \text{const}$ inside the layer. Using submodel 2.22 as an example, we show how these functions can be used to control the geometry of gas motion.

For the case of motion described by submodel 2.22, we consider a set of particles lying on a straight line parallel to the λ axis at the initial time $t = 1$: $y = y_0$, $z = z_0$. The deformation of this straight line at $t > 1$ is defined by the functions $V_0(x_0)$ and $W_0(x_0)$. In particular, these functions can be chosen such that at the collapse moment t_* ($M(t_*) = 0$), the particles lying at the initial time $t = 1$ on the straight line $y = y_0$, $z = z_0$ form a previously specified curve $y = h(s)$, $z = k(s)$ on the collapse plane $x = 0$:

$$V_0(x_0) = \frac{h(x_0) - t_* y_0}{t_* - 1}, \qquad W_0(x_0) = \frac{k(x_0) - t_* z_0}{t_* - 1}. \tag{4.1}$$

Remark 2. *Under mapping by formulas* (3.5) *with the functions* V_0 *and* W_0 *from* (4.1)*, the complete preimage of a point occupying a position* $y = y_*$, $z = z_*$ *on the collapse plane* $x = 0$ *at* $t = t_*$ *is given by*

$$y = y_0 + (y_* - h(x))/t_*, \qquad z = z_0 + (z_* - k(x))/t_*, \qquad -\infty < x < +\infty.$$

5. Analysis of the Equation for the Function M. To construct classes of exact solutions of Eqs. (3.2), (3.4), (3.7), and (3.9), group classification with respect to the adiabatic exponent γ was performed for each equation. From the viewpoint of physics, we considered only values $\gamma > 1$. The symmetries found were used for full or partial integration of the equations.

Submodel 2.21. The nontrivial operator $(t^2 + 1)\partial_t + tM\partial_M$ is admitted only if $\gamma = 5/3$. In this case, by the substitution $M = N(\tau)\sqrt{t^2 + 1}$, where $\tau = \arctan t$, and single integration, Eq. (3.2) is reduced to the form

$$\dot{N}^2 = \beta - 3\alpha N^{-2/3} - N^2 \equiv h(N), \qquad \beta = \text{const}. \tag{5.1}$$

The initial condition follows from the condition for M: $N(0) = M(0) = 1$. The behavior of the function $h(N)$ on the right side of (5.1) is determined by the sign of α. If $\alpha < 0$, the function $h(N)$ decreases monotonically on the semiaxis $N > 0$ and has a single root $N_{\max} > 0$. For $\alpha > 0$, the function h is positive only if $\beta > 4\alpha^{3/4}$. In this case there are two positive simple roots $h(N_l) = h(N_r) = 0$, such that $h'(N_l) > 0$, $h'(N_r) < 0$, $h(N) > 0$, and $N \in (N_l, N_r)$. The solution of Eq. (5.1) is written in terms of the function

$$J(a,b) = \int\limits_a^b \frac{d\eta}{\sqrt{\beta - 3\alpha\eta^{-2/3} - \eta^2}}.$$

If $\alpha > 0$ and $\beta > 4\alpha^{3/4}$, Eq. (5.1) has solutions which are periodic over τ [8]. The half-period of the solution is $T = J(N_l, N_r)$. The dependence $N(\tau)$ at $\tau \in [0, T]$ is given implicitly by the relation $J(N_l, N) = \tau$ and is extended to all values of τ as an even periodic function. To satisfy the initial condition $N(0) = 1$, the function obtained must be displaced over the τ axis by the quantity $J(N_l, 1)$ or $J(1, N_r)$, depending on the sign of the derivative $\dot{M}(0)$. A plot of the function $N(\tau)$ is shown in Fig. 1 (solid curve).

For $\alpha < 0$, the solution is constructed similarly. On the interval $\tau \in [0, \tau_0]$ with $\tau_0 = J(0, N_{\max})$, the function $N(\tau)$ was defined implicitly by the equation $J(0, N) = \tau$. Then, the function is continued evenly over the entire interval $[-\tau_0, \tau_0]$ and is shifted on the τ axis in order to satisfy the initial condition. A smooth solution exists only on the finite interval $\tau \in (0, \tau_*)$ and $\dot{N}(\tau_*) = \infty$ (dashed curve in Fig. 1). The corresponding function $M(t)$ vanishes at finite t if $\tau_* < \pi/2$ or, otherwise, it increases without bound.

Submodels 2.22, 2.24, and 2.25. Zaitsev and Polyanin [9] give a large number of exact solutions of the generalized Emden–Fowler equation, whose particular cases are Eqs. (3.4), (3.7), and (3.9). For an arbitrary γ, they admit only the dilation operator $X_1 = t\partial_t + \sigma M\partial_M$, where $\sigma = (4-2\gamma)(1+\gamma)^{-1}$ for (3.4) and $\sigma = (3-\gamma)(1+\gamma)^{-1}$ for (3.7). For an arbitrary γ, in addition to the operator X_1 for $\sigma = 2(1+\gamma)^{-1}$, Eq. (3.9) admits translation over time $X_2 = \partial_t$. For special adiabatic exponents $\gamma = 5/3$ for (3.4), $\gamma = 2$ for (3.7), and $\gamma = 3$ for (3.9), the admitted algebra is dilated by the projective operator $X_3 = t^2\partial_t + tM\partial_M$.

For an arbitrary γ, the substitution $M = t^\sigma N(\tau)$, where $\tau = \ln|t|$, reduces Eqs. (3.4) and (3.7) to the autonomous form

$$\ddot{N} + (2\sigma - 1)\dot{N} + \sigma(\sigma - 1)N = \alpha N^{-\gamma}.$$

Equation (3.9) is integrated:

$$\dot{M}^2 = \beta - 2\alpha(\gamma - 1)^{-1}M^{1-\gamma} \qquad (\beta = \text{const}). \tag{5.2}$$

For the special adiabatic exponents, the substitution $M = tN(\tau)$, where $\tau = t^{-1}$, and single integration reduce Eqs. (3.4) and (3.7) to the equations

$$(\dot{N})^2 = \beta - 3\alpha N^{-2/3} \qquad (\beta = \text{const}) \tag{5.3}$$

and

$$(\dot{N})^2 = \beta - 2\alpha N^{-1} \qquad (\beta = \text{const}), \tag{5.4}$$

respectively. If $\gamma = 3$, Eq. (3.9) is explicitly integrated:

$$\beta M^2 = \beta^2(t - t_0)^2 + \alpha. \tag{5.5}$$

Relation (5.5) defines families of ellipses and hyperbolas on the plane (t, M). Sedov [3] classifies the types of solutions of Eqs. (5.2)–(5.4) depending on the sign of the constants α and β. There are three types of $N(\tau)$ [in Eq. (5.2), it is assumed that $M \equiv N$ and $\tau \equiv t$]:

1) $\alpha > 0$ and $\beta > 0$. In this case, the function $N(\tau)$ is convex downward ($\ddot{N} > 0$) and has a single minimum point $N = N_{\min}$ at $\tau = \tau_*$. As $\tau \to \pm\infty$, the derivative tends to the constant value $\dot{N} \to \pm\sqrt{\beta}$. A plot of the function is given in Fig. 2 (bold curve).

2) $\alpha < 0$ and $\beta > 0$. In this case, the function $N(\tau)$ is convex upward ($\ddot{N} < 0$) and there is a single collapse point $\tau = \tau_*$, at which $N(\tau_*) = 0$ and $\dot{N}(\tau_*) = \infty$. In this case, $\dot{N} \to \pm\sqrt{\beta}$ as $\tau \to \pm\infty$ (thin curve in Fig. 2).

3) $\alpha < 0$ and $\beta < 0$. In this case, pulsations with collapse take place; the function $N(\tau)$ is convex upward and does not exceed the value of $N_{\max}$. A smooth solution exists only in a finite interval over τ, on whose boundary, $N(\tau_*) = 0$ and $\dot{N}(\tau_*) = \infty$. A plot of the function is given in Fig. 2 (dashed curve).

6. Submodels of the Second Type. *Submodel* 2.8. In the cylindrical coordinates (x, r, θ), the solution is written as

$$u_c = V + x/t, \qquad v_c = U, \qquad w_c = W, \qquad \lambda = r.$$

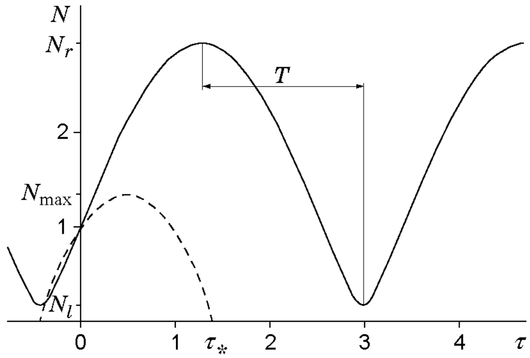

Fig. 1

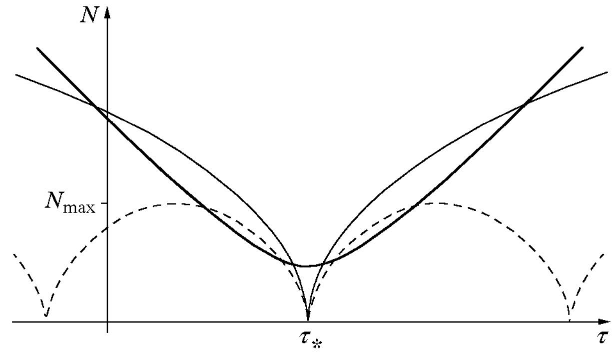

Fig. 2

Here and below, u_c, v_c, and w_c are the axial, radial, and tangential velocity components, respectively. In this submodel, $b = 1$, $a_1 = W^2/\lambda$, $a_2 = -V/t$, $a_3 = -UW/\lambda$, and $a_4 = -(U/\lambda + 1/t)$. Integration of (2.1) taking into account (2.2) yields

$$V = V_0(\xi)/t, \quad W = W_0(\xi)/M(t), \quad \rho = f(\xi)/(M^2 t), \quad p = P(\xi)/(M^{2\gamma} t^\gamma). \tag{6.1}$$

As above, the functions V_0, W_0, f, and P are arbitrary. In separation of variables in Eq. (2.4) there are two cases by virtue of the Lemma. If $\gamma = 3/2$, the function M is defined explicitly: $M = \alpha\sqrt{t}$. In this case, a constraint is imposed on the arbitrary functions $W_0^2(\xi) = \alpha\xi P'(\xi)/f(\xi) - \alpha^4\xi^2/4$. In the second case, γ is arbitrary, and the function M satisfies the equation

$$\ddot{M}M^3 + \alpha t^{1-\gamma} M^{4-2\gamma} = \beta^2, \qquad \alpha, \beta = \text{const}. \tag{6.2}$$

For this submodel, $P'(\xi) = \alpha\xi f(\xi)$ and $W_0 = \beta\xi$. If $t_0 = 1$, the particle trajectories are defined by the formulas

$$x = (x_0 + V_0(r_0))t - V_0(r_0), \qquad r = r_0 M(t), \qquad \theta = \theta_0 + \frac{W_0(r_0)}{r_0} \int \frac{dt}{M^2}.$$

Submodel 2.9. The solution representation is

$$u_c = V + \beta\theta, \qquad v_c = U, \qquad w_c = W, \qquad \lambda = r.$$

For this submodel, $b = 1$, $a_1 = W^2/\lambda$, $a_2 = -\beta W/\lambda$, $a_3 = -UW/\lambda$, and $a_4 = -U/\lambda$. Integration of (2.1) taking into account (2.2) yields

$$V = -\frac{\beta W_0(\xi)}{\xi} \int \frac{dt}{M^2} + V_0(\xi), \quad W = \frac{W_0(\xi)}{M(t)}, \quad \rho = \frac{f(\xi)}{M^2}, \quad p = \frac{P(\xi)}{M^{2\gamma}}. \tag{6.3}$$

In separation of variables in Eq. (2.4) there are two cases. In the first case, $M \equiv 1$, $U = 0$, $V = -\beta W_0(r)t + V_0(r)$, $W = W_0(r)$, $\rho = f(r)$, $p = P(r)$, and $P'(r) = f(r)W_0(r)$. This solution defines an axisymmetrical shear flow. The trajectory of each particle is a spiral wound around a circular cylinder. The second case presents more complex motions of a gas. The function M satisfies the equation

$$\ddot{M}M^3 + \alpha M^{4-2\gamma} = \sigma^2, \qquad \alpha, \sigma = \text{const}. \tag{6.4}$$

In addition, $P'(\xi) = \alpha\xi f(\xi)$ and $W_0 = \sigma\xi$. The particle trajectories with $t_0 = 0$ are defined by the formulas

$$x = x_0 + (\beta\theta_0 + V_0(r_0))t, \qquad r = r_0 M(t), \qquad \theta = \theta_0 + \frac{W_0(r_0)}{r_0} \int \frac{dt}{M^2}.$$

Submodel 2.10. Mustaev and Khabirov [10] analyzed this submodel. The solution representation as

$$u_c = V - (\theta - x)/t, \qquad v_c = U, \qquad w_c = W, \qquad \lambda = r.$$

For this submodel, $b = 1$, $\tilde{a}_1 = W^2/\lambda$, $\tilde{a}_2 = (W - \lambda V)/(t\lambda)$, $\tilde{a}_3 = -UW/\lambda$, and $\tilde{a}_4 = -(U/\lambda + 1/t)$. Integrating (2.1) with allowance for (2.2), we have

$$V = \frac{W_0(\xi)}{t\xi} \int \frac{dt}{M^2} + \frac{V_0(\xi)}{t}, \quad W = \frac{W_0(\xi)}{M(t)}, \quad \rho = \frac{f(\xi)}{M^2 t}, \quad p = \frac{P(\xi)}{M^{2\gamma} t^\gamma}. \tag{6.5}$$

The key equation (2.4) in this submodel coincides with that in submodel 2.8. Therefore, in this submodel, there are also two possible cases for the function M and the same constraints are imposed on W_0, P, and f. The particle trajectories with $t_0 = 1$ are defined by the formulas

$$x = x_0 t + (\theta_0 - V_0(r_0))(1-t), \quad r = r_0 M(t), \quad \theta = \theta_0 + \frac{W_0(r_0)}{r_0}\int \frac{dt}{M^2}.$$

7. Description of Motion. The motion of particles for solutions of the second type can be represented as follows. Along the trajectory, the axial coordinate x depends linearly on time t in all the three solutions. Hence, the trajectory of each particle can be parametrized by the x coordinate instead of t. Then, the law of variation of the radial coordinate of a particle defines a surface of revolution with generatrix $M(x)$ in the space of (x, r, θ). The particle trajectory is a spiral wound around this surface according to the law of variation of θ. We note that if $M(t_*) = 0$ for a certain t_*, all particles arrive at the Ox axis at time t_* during this motion.

To describe the motion of gas particles in the aggregate, we consider a gas enclosed in a cylinder $r = r_0 M(t)$. The pressure on the cylinder surface is the same at all points and is given by Eqs. (6.1), (6.3), and (6.5), where $P(\xi) = P(r_0) = \text{const}$. If the function $M(t)$ increases, the gas is rarefied. At the same time, as $M(t) \to 0$, the gas cylinder compresses and transforms into a straight line, and the density and pressure increase to infinity. The gas particles move along the cylinder axis and rotate about it. If the integral $\displaystyle\int M^{-2} dt$ converges, the particles perform a finite number of revolutions around the axis; otherwise, they perform an infinite number of revolutions.

The function $V_0(\xi)$ defines a preimage of a point on the collapse manifold. Thus, in submodel 2.8, the preimage of the point $x = x_*$ at $t = t_*$ $[M(t_*) = 0]$ is the surface of revolution

$$x = t_*^{-1}(x_* + (1 - t_*)V_0(r)), \qquad 0 \leqslant r < +\infty.$$

In submodels 2.9 (for $\beta \neq 0$) and 2.10, a solution that is continuous over the entire space does not exist because the velocity component u depends linearly on the polar angle θ. At the same time, motion with discontinuity along an infinite surface with the edge on the Ox axis is possible. The initial position of the discontinuity surface is determined arbitrarily in the coordinates x_0, r_0, and θ_0 for $t = 1$. Further, its form and position are calculated by the formulas that define particle trajectories.

Remark 3. *In all the solutions constructed, the functions $V_0(\xi)$ and $W_0(\xi)$ can have discontinuities of the first kind. These correspond to solutions with a discontinuity along the surfaces $\lambda = \lambda_0 M(t)$.*

8. Analysis of the Equation for the Function M. *Submodels* 2.8 and 2.10. For an arbitrary γ, Eq. (6.2) does not admit point transformations.

If $\gamma = 5/3$, the projective operator $X = t^2\partial_t + tM\partial_M$ is admitted. In this case, the substitution $M = tN(\tau)$, where $\tau = t^{-1}$, and single integration reduce Eq. (6.2) to the form

$$(\dot{N})^2 = \sigma + 3\alpha N^{-4/3}/2 - \beta^2 N^{-2} \equiv h(N, \sigma), \qquad \sigma = \text{const}. \tag{8.1}$$

Equation (8.1) has solutions that are periodic in τ. To prove this, we need to establish the existence of an interval $[N_1,\, N_2]$ such that

$$h(N_1, \sigma) = h(N_2, \sigma) = 0, \qquad h_N(N_1, \sigma) > 0, \qquad h_N(N_2, \sigma) < 0 \tag{8.2}$$

($h_N \equiv \partial h/\partial N$) [8]. The existence of such an interval is ensured by the existence of a point with the coordinates $N_* = (\beta^2/\alpha)^{3/2}$ and $\sigma_* = -2\alpha/(9\beta^2)$ for $\alpha > 0$, at which

$$h(N_*, \sigma_*) = 0, \qquad h_N(N_*, \sigma_*) = 0,$$

$$h_{NN}(N_*, \sigma_*) = -(4/3)(\beta^2/\alpha)^{-6} < 0, \qquad h_\sigma(N_*, \sigma_*) = 1 > 0.$$

The periodic function $N(\tau)$ takes values $0 < N_1 \leqslant N \leqslant N_2 \leqslant \infty$. The function $M(t)$ is plotted between the straight lines $M = N_1 t$ and $M = N_2 t$ and performs an infinite number of oscillations between them in the vicinity of $t = 0$. In this case, $M(0) = 0$ and the derivative $M'(t)$ has a discontinuity of the second kind at the point $t = 0$ (Fig. 3).

If $\gamma = 3/2$, Eq. (6.2) admits the dilation operator $X = 2t\partial_t + M\partial_M$. The substitution $M = N(\tau)\sqrt{t}$, where $\tau = \ln|t|$, and single integration reduce Eq. (6.2) to the form

$$(\dot{N})^2 = \sigma + N^2/4 + 2\alpha N^{-1} - \beta^2 N^{-2} \equiv h(N, \sigma), \qquad \sigma = \text{const}. \tag{8.3}$$

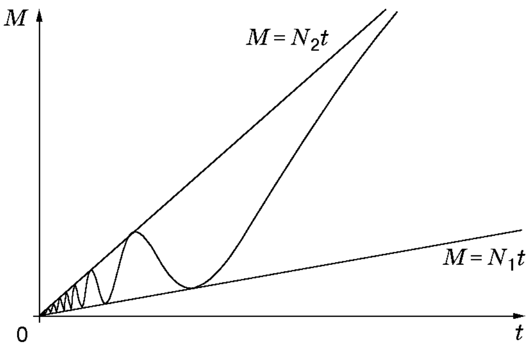

Fig. 3

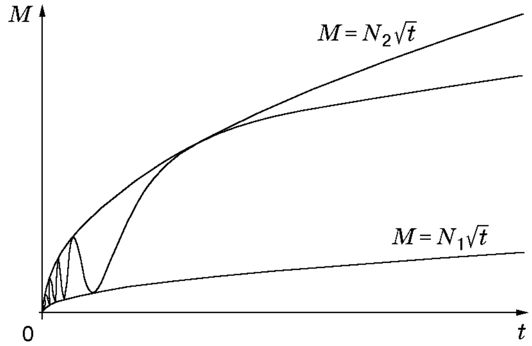

Fig. 4

To prove the existence of periodic solutions of Eq. (8.3), we prove the existence of an interval $[N_1, N_2] \subset (0, \infty)$ that satisfies conditions (8.2). The zeros of the function $h(N, \sigma)$ coincide with the zeros of the polynomial $Q(N) = (1/4)N^4 + \sigma N^2 + 2\alpha N - \beta^2$. Using the Sturm theorem, we find that for $\alpha > 0$ and $\sigma < 0$, the polynomial $Q(N)$ has one negative root and two positive roots. Because $Q(N) \to +\infty$ as $N \to +\infty$, the first two positive roots of $Q(N)$ are the desired boundaries of the interval $[N_1, N_2]$. The periodic solution of $N(\tau)$ corresponds to a function $M(t)$ whose plot is located between curves of $M = N_1\sqrt{t}$ and $M = N_2\sqrt{t}$ and performs an infinite number of oscillations between these curves in the vicinity of $t = 0$. In this case, $M(0) = 0$ and $M'(0) = +\infty$ (Fig. 4).

For $\gamma = 2$, Eq. (6.2) admits the operator $(4\beta^2 t^2 - 4\alpha t)\partial_t + (4\beta^2 t - \alpha)M\partial_M$. The substitution $M = t^{1/4}(\alpha - t\beta^2)^{3/4} N(\tau)$, where $\tau = \ln|\beta^2 - \alpha/t|$, reduces Eq. (6.2) to the autonomous form

$$8\alpha^2 N^3(2\ddot{N} + \dot{N}) = 3\alpha^2 N^4 - 16.$$

Submodel 2.9. For an arbitrary γ, Eq. (6.4) admits only translation over time $X = \partial_t$. Single integration of this equation yields

$$\dot{M}^2 = \delta - \sigma^2 M^{-2} - \alpha M^{2-2\gamma}/(1-\gamma) \equiv h(M, \delta). \tag{8.4}$$

For $\alpha > 0$, there exists a point with the coordinates $M_* = k^{1/(4-2\gamma)}$ and $\delta_* = \gamma\sigma^2(\gamma - 1)^{-1} k^{1/(\gamma-2)}$ $(k = \sigma^2/\alpha)$, at which

$$h(M_*, \delta_*) = 0, \qquad h_M(M_*, \delta_*) = 0,$$

$$h_{MM}(M_*, \delta_*) = (\gamma - 2)\alpha^{-1} k^{\gamma/(\gamma-2)}, \qquad h_\delta(M_*, \delta_*) = 1 > 0.$$

Thus, if $1 < \gamma < 2$, the conditions of existence of a periodic solution are satisfied for Eq. (8.4). If $\beta = 0$, this solution is a "gas pendulum" [11]. If $\beta \neq 0$, a solution is obtained that has so-called phase periodicity over time [8] (after a lapse of a period, a general flow pattern is restored but the gas particles do not return to the initial position). If $\gamma = 2$, Eq. (6.4) is integrated explicitly:

$$\delta M^2 = \delta^2(t - t_0)^2 + \sigma^2 - \alpha. \tag{8.5}$$

Equation (8.5) defines the families of ellipses and hyperbolas on the plane (t, M).

9. Submodels of the Third Type. In submodels of the third type, we have to deviate from the canonical form of their representation and choose invariant functions differently.

Submodel 2.20. The solution representation is

$$u = U - \frac{(\beta\sigma - \alpha t)(\tau V + tW)}{(t^2 - \sigma\tau)^2} + \frac{(\alpha\tau - \beta t)(tV + \sigma W)}{(t^2 - \sigma\tau)^2},$$

$$v = \frac{\tau V + tW}{t^2 - \sigma\tau} + \frac{ty - \tau z}{t^2 - \sigma\tau}, \qquad w = -\frac{tV + \sigma W}{t^2 - \sigma\tau} + \frac{tz - \sigma y}{t^2 - \sigma\tau},$$

$$\rho = \rho(t, \lambda), \qquad p = p(t, \lambda), \qquad \lambda = x + \frac{\beta\sigma - \alpha t}{t^2 - \sigma\tau}\, y + \frac{\alpha\tau - \beta t}{t^2 - \sigma\tau} z.$$

The equations of the submodel have the form

$$DU+\frac{t^4+(\alpha^2+\beta^2-2\sigma\tau)t^2-2\alpha\beta(\sigma+\tau)t+\beta^2\sigma^2+(\alpha^2+\sigma^2)\tau^2}{(t^2-\sigma\tau)^2\rho}\,p_\lambda$$

$$=-2\frac{\beta t^3-3\alpha\tau t^2+3\beta\sigma\tau t-\alpha\sigma\tau^2}{(t^2-\sigma\tau)^3}\,V-2\,\frac{-\alpha t^3+3\beta\sigma t^2-3\alpha\sigma\tau t+\beta\sigma^2\tau}{(t^2-\sigma\tau)^3}\,W,$$

$$DV+\frac{\beta t^2+\alpha(\sigma-\tau)t-\beta\sigma^2}{(t^2-\sigma\tau)\rho}\,p_\lambda=0,\qquad D=\partial_t+U\partial_\lambda,$$

$$DW+\frac{-\alpha t^2+\beta(\sigma-\tau)t+\alpha\tau^2}{(t^2-\sigma\tau)\rho}\,p_\lambda=0,$$

$$D\rho+\rho U_\lambda=-\frac{2t\rho}{t^2-\sigma\tau},\qquad Dp+A(p,\rho)U_\lambda=-A(p,\rho)\frac{2t}{t^2-\sigma\tau}.$$

Below, we assume that the gas satisfies the polytropic equation of state $A(p,\rho)=\gamma p$. After integration in the Lagrangian coordinates t and ξ, we obtain

$$\begin{gathered}\rho=\frac{f(\xi)}{M(t^2-\sigma\tau)},\qquad p=\frac{P(\xi)}{M^\gamma(t^2-\sigma\tau)^\gamma},\\ V=(\beta J_2+\alpha(\sigma-\tau)J_1-\beta\sigma^2J_0)P'(\xi)/f(\xi)+V_0(\xi),\\ W=(-\alpha J_2+\beta(\sigma-\tau)J_1+\alpha\tau^2J_0)P'(\xi)/f(\xi)+W_0(\xi).\end{gathered}\tag{9.1}$$

Here $J_0=\displaystyle\int\frac{dt}{(t^2-\sigma\tau)^\gamma M^\gamma}$, $J_1=\displaystyle\int\frac{t\,dt}{(t^2-\sigma\tau)^\gamma M^\gamma}$, and $J_2=\displaystyle\int\frac{t^2\,dt}{(t^2-\sigma\tau)^\gamma M^\gamma}$. In the remaining equation

$$\ddot M\xi+\frac{t^4+(\alpha^2+\beta^2-2\sigma\tau)t^2-2\alpha\beta(\sigma+\tau)t+\beta^2\sigma^2+(\alpha^2+\sigma^2)\tau^2}{(t^2-\sigma\tau)^{1+\gamma}M^\gamma}\,\frac{P'(\xi)}{f(\xi)}$$

$$+2\,\frac{\beta t^3-3\alpha\tau t^2+3\beta\sigma\tau t-\alpha\sigma\tau^2}{(t^2-\sigma\tau)^3}\,V+2\,\frac{-\alpha t^3+3\beta\sigma t^2-3\alpha\sigma\tau t+\beta\sigma^2\tau}{(t^2-\sigma\tau)^3}\,W=0,$$

it is necessary to substitute the expressions for V and W from (9.1) and separate the variables in it according to the Lemma. In this case, we obtain intricate formulas, which are omitted here.

Submodel 2.23. The solution representation is

$$u=U(\lambda)+z+\alpha V(\lambda)+tW(\lambda),\qquad v=V(\lambda),\qquad w=W(\lambda),$$

$$p=p(\lambda),\qquad \rho=\rho(\lambda),\qquad \lambda=x-\alpha y-tz.$$

The equations of the submodel have the form

$$U_t+UU_\lambda+(t^2+\alpha^2+1)p_\lambda/\rho=-2W,$$

$$V_t+UV_\lambda-\alpha p_\lambda/\rho=0,\qquad W_t+UW_\lambda-tp_\lambda/\rho=0,$$

$$\rho_t+U\rho_\lambda+\rho U_\lambda=0,\qquad p_t+Up_\lambda+A(p,\rho)U_\lambda=0.$$

Next, we consider a polytropic gas [$A(p,\rho)=\gamma p$]. In the Lagrangian coordinates t and ξ, integration yields

$$\rho=\frac{f(\xi)}{M},\quad p=\frac{P(\xi)}{M^\gamma},\quad V=\frac{\alpha P'(\xi)}{f(\xi)}\int\frac{dt}{M^\gamma}+V_0(\xi),\quad W=\frac{P'(\xi)}{f(\xi)}\int\frac{t\,dt}{M^\gamma}+W_0(\xi).$$

The remaining equation of the submodel takes the form

$$\ddot M\xi+\left(\frac{t^2+\alpha^2+1}{M^\gamma}+2\int\frac{t\,dt}{M^\gamma}\right)\frac{P'(\xi)}{f(\xi)}+2W_0(\xi)=0.$$

According to the Lemma there are two possible cases:

1) $\gamma=2$, $M=t^2+\alpha^2+1$, and $W_0=-\xi$;

2) $\ddot M+a\Big(\dfrac{t^2+\alpha^2+1}{M^\gamma}+2\displaystyle\int\frac{t\,dt}{M^\gamma}\Big)+2b=0$, $P'(\xi)=a\xi f(\xi)$, and $W_0=b\xi$.

Conclusions. A new class of exact solutions describing three-dimensional motions of a polytropic gas is constructed. The solutions describe continuous dispersion and infinite compression of the gas over finite or infinite time intervals. These solutions were derived for the evolutionary invariant submodels of rank two. In the solutions obtained, the velocity depends linearly on certain spatial coordinates. The solutions contain three arbitrary functions of one variable. Integration of the submodels reduces to solution of a second-order ordinary differential equation. All submodels are divided into two types. Submodels of the first type describe compression or rarefaction of a flat gas layer. The particle trajectories are plane curves; the slope and position of the plane in space are determined by the initial position of the particle. In motion with density collapse, a preimage of a point on the collapse plane is a certain spatial curve, whose initial configuration can be chosen rather arbitrarily by virtue of the existing arbitrariness of the solution. Submodels of the second type describe the compression or rarefaction of the gas filling a cylinder whose radius varies with time. Moving along the cylinder axis, the particles perform a finite or infinite number of revolutions around the axis. In the case of motion with density collapse, the cylinder is compressed and transformed into a straight line. The existence of oscillatory regimes of gas motion is shown.

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